# Functional Anaysis for Engineers

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# Contents



# 1 Vector Spaces

**Definition 1.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then a set $V$ is called $\mathbb{K}$-vector space with (operators) $+ : V \times V \to V$ and $\cdot : V \times \mathbb{K} \to V$ if the following holds*

*a)* $(V,+)$ *is a commutative group. E.g. this means:*
$(v+0=v,\ (v+w)+g=v+(w+g),\ v+w=w+v,\ v+(-v)=0)$.

*b)* *i)* $(a+b)\cdot\lambda = a\cdot\lambda + b\cdot\lambda$ *for all* $a,b\in V,\quad \lambda\in\mathbb{K}$

*ii)* $a\cdot(\lambda+\rho) = a\cdot\lambda + a\cdot\rho$ *for all* $a\in V,\quad \rho,\lambda\in\mathbb{K}$

*iii)* $a\cdot(\lambda\rho) = (a\cdot\lambda)\cdot\rho$ *for all* $a\in V,\quad \rho,\lambda\in\mathbb{K}$

*iv)* $a\cdot 1 = a$ *for all* $a\in V$

**Example 1.**

a) Example in three dimensions: $\underline{\mathbb{K}}^3$

$$\begin{pmatrix}5\\3\\2\end{pmatrix}\cdot 2 = \begin{pmatrix}10\\6\\4\end{pmatrix}$$

$$\begin{pmatrix}5\\3\\2\end{pmatrix}+\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}6\\5\\6\end{pmatrix}$$

Example in the n-dimensional space $\underline{\mathbb{K}}^n$

$$(a_i)_{i=1,\dots,n} + (b_i)_{i=1,\dots,n} = (a_i+b_i)_{i=1,\dots,n}$$
$$(a_i)_{i=1,\dots,n}\lambda = (a_i\lambda)_{i=1,\dots,n}$$

b) Let $A,B$ two sets (not empty)

$$\mathcal{F}(A,B) := \{f : A\to B \mid \text{f is a function}\}.$$

Then $\mathcal{F}(A,\mathbb{R})$ is a vector space with operators

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x),\\ (f\cdot\lambda)(x) &:= f(x)\cdot\lambda\end{aligned}$$

for every $f,g\in\mathcal{F}(A,\mathbb{R})$.

c) Let $\Omega\subset\mathbb{R}^n$ be an open subset. Then $\mathcal{C}_{\mathbb{R}}(\Omega) = \{f:\Omega\to\mathbb{R}\mid f \text{ is continuous}\}$ is a subspace of vector space $\mathcal{F}(\Omega,\mathbb{R})$.

d) Let $\mathcal{R}_{\mathbb{K}}([a,b])$ be the set of Riemann integrable functions

$$\mathcal{R}_{\mathbb{K}}([a,b])\subset\mathcal{F}([a,b],\mathbb{K}).$$

# 2 Normed Vector Spaces

**Definition 2.** *Let $V$ be a $K$ vector space. A mapping $p : V \to [0, \infty[$ is called semi-norm if the following holds:*

*a) $p(\lambda x) = |\lambda| p(x) \qquad \forall \lambda \in \mathbb{K}, x \in V$,*

*b) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$ (triangle inequality).*

*$p$ is called norm if $p(x) = 0 \Rightarrow x = 0$.*
*A vector space $V$ with norm $\| \cdot \|$ is called normed vector space.*
*Usually we write*

$$\begin{array}{ll} \| \cdot \| & \text{for norms and} \\ | \cdot | & \text{for semi-norms.} \end{array}$$

**Example 2.**

a) The Euclidian norm on $\mathbb{R}^n$ is defined by

$$\|(a_i)_{i=1,\dots,n}\|_2 := \sqrt{\sum_{i=1}^{n} |a_i|^2}.$$

b) Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded subset (e.g. $\Omega = [a, b]$). The maximum norm on $\mathcal{C}_{\mathbb{R}}(\Omega)$ is defined by

$$\|f\|_\infty := \max_{x \in \Omega} |f(x)| \quad \text{for f} \in \mathcal{C}_{\mathbb{R}}(\Omega).$$

<u>Proof:</u> $\| \cdot \|_\infty$ is a norm on $C_{\mathbb{R}}(\Omega)$ . Since $\Omega$ closed and bounded, we get $\max_{x \in \Omega} |f(x)| \in [0, \infty[$ by elementary analysis. Furthermore, there exists a $x_0$ such that $\max_{x \in \Omega} |f(x)| = |f(x_0)|$. This implies that:

$$\begin{array}{rcl} |\lambda| \, \|f\|_\infty = |\lambda| \, |f(x_0)| & = & |\lambda f(x_0)| \\ & \leq & \max_{x \in \Omega} |\lambda f(x)| = \|\lambda f\|_\infty. \end{array}$$

Assume that $\lambda \neq 0$. Then, we get

$$\begin{array}{lcl} |\frac{1}{\lambda}| \, \|\lambda f\|_\infty & \leq & \|\frac{1}{\lambda} \lambda f\|_\infty = \|f\|_\infty \\ \|\lambda f\|_\infty & \leq & |\lambda| \, \|f\|_\infty \leq \|\lambda f\|_\infty \\ & \Downarrow & \\ & \|\lambda f\|_\infty = & |\lambda| \, \|f\|_\infty. \end{array}$$

The triangle inequality follows by

$$\begin{aligned} \|f+g\|_\infty &= |f(x_0)+g(x_0)| \le |f(x_0)|+|g(x_0)| \\ &\le \max_{x\in\Omega}|f(x)| + \max_{x\in\Omega}|g(x)| \\ &= \|f\|_\infty + \|g\|_\infty . \end{aligned}$$

At least, let us prove positive definiteness. Assume $\|f\|_\infty = 0$. Then,

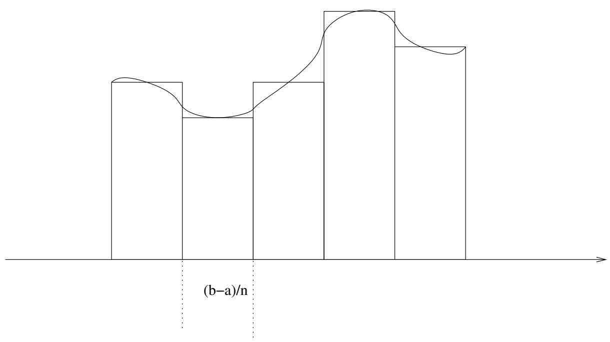
$$\begin{aligned} \max_{x\in\Omega}|f(x)| = 0 \quad &\Rightarrow \quad |f(x)| = 0 \qquad \forall x\in\Omega \\ &\Rightarrow \quad f(x) = 0 \qquad \forall x\in\Omega. \end{aligned}$$

Therefore, $f$ is the zero-element in $C_{\mathbb{R}}(\Omega)$. □

**Example 3.** Take $\mathcal{R}([a,b])$. If $f$ is Riemann-integrable, then

$$\int_a^b f(x)\,dx \;:= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n f\left(a + (i-\frac{1}{2})\frac{b-a}{n}\right)$$

converges.




Let us define the space

$$\mathcal{R}^2([a,b]) := \{f : [a,b] \to \mathbb{R} \mid |f|^2 \in \mathcal{R}([a,b])\}$$

and the semi-norm:

$$|f|_{L^2[a,b]} := \sqrt{\int_a^b |f(x)|^2 dx}.$$

This semi-norm can be obtained by the limit of the Euclidian norm

$$\|(a_i)_{i\in 1,\dots,n}\|_2 := \sqrt{\sum_{i=1}^n |a_i|^2}.$$

Let $f \in \mathcal{R}^2([a,b])$ be Riemann-integrable. Then, we get

$$\begin{aligned}
|f|^2_{L^2([a,b])} &:= \int_a^b |f(x)|^2 dx \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left| f(a+(i-\frac{1}{2})\frac{b-a}{n}) \right|^2 \\
&= \lim_{n\to\infty} \left( \frac{b-a}{n} \left\| \left( f(a+(i-\frac{1}{2})\frac{b-a}{n}) \right)_{i=1,\dots,n} \right\|^2_{\mathbb{R}^n} \right)
\end{aligned}$$

But $|f|_{L^2([a,b])}$ is semi-norm! For example, consider the function:

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{b+a}{2} \\ 0 & \text{else} \end{cases} \Rightarrow |f|_{L([a,b])} = 0.$$

## 2.1 Quotient Construction

On $\mathcal{R}^2([a,b])$ we define the relation

$$f \equiv g :\Longleftrightarrow |f-g|_{L^2([a,b])} = 0.$$

The equivalence classes of $\mathcal{R}^2([a,b])$ with respect to $\equiv$ define a new important normed vector space:

$$\begin{aligned}
R^2([a,b]) := \mathcal{R}^2([a,b]) / &\equiv \\
&= \left\{ [f] := \left\{ g \in \mathcal{R}^2([a,b]) \;\middle|\; g \equiv f \right\} \;\middle|\; f \in \mathcal{R}^2([a,b]) \right\}.
\end{aligned}$$

**Lemma 1.** *$R^2([a,b])$ is a vector space with operators:*

$$\begin{aligned}
[g] + [w] &:= [g+w] \quad \forall [g],[w] \in R^2([a,b]) \quad \textit{and} \\
[g] \cdot \lambda &:= [g\lambda] \quad \forall \lambda \in \mathbb{R},\ \forall [g] \in R^2([a,b]).
\end{aligned}$$

*$\|[g]\|_{L^2} := |g|_{L^2}$ is a norm on $R^2([a,b])$.*

Proof: Let us show that the operation $+$ is well defined. Assume that

$$[g_1] = [g_2], [w_1] = [w_2].$$

Then, we get $[g_1 + w_1] = [g_2 + w_2]$, since

$$\begin{aligned}
[g_1 + w_1] = [g_2 + w_2] &\Longleftarrow |(g_1 + w_1) - (g_2 + w_2)|_{L^2} = 0 \\
&\Longleftarrow |(g_1 - g_2) + (w_1 - w_2)|_{L^2} = 0 \\
&\Longleftarrow |(g_1 - g_2) + (w_1 - w_2)|_{L^2} \leqslant |g_1 - g_2|_{L^2} + |w_1 - w_2|_{L^2} = 0.
\end{aligned}$$

Analogously, it can be proved that $\cdot$ is well defined. To prove the axioms of a vector space and axioms of a semi-norm is left to the reader. By construction, $\|[g]\|_{L^2}$ is a norm on $L^2([a,b])$.

□

Instead of $[f]$, we often just write $f$. Then, we can denote the space $R^2([a,b])$ by

$$R^2([a,b]) := \left\{ f : [a,b] \to \mathbb{R} \;\Big|\; \int_a^b |f(x)|^2 \mathrm{dx} < \infty \right\}$$

with norm

$$\|f\|_{L^2} := (\int_a^b |f(x)|^2 \mathrm{dx})^{\frac{1}{2}}.$$

## 2.2 Topology of a Normed Vector Space

**Definition 3.** *"Topology". Let $A$ be a set. A collection $U$ of subsets of $A$ is called topology on $A$, if the following conditions hold:*

*i)* $\emptyset \in U, A \in U$

*ii)* $O, P \in U \Rightarrow O \cap P \in U$

*iii) Let $I$ be a subset of $U$. Then* $\bigcup_{O \in I} O \in U$.

*The sets $O \in U$ are called open sets of the topology.*

**Example 4.** In $\mathbb{R}$, we get

$$\begin{aligned}
]a,b[ \;&:= \{x \in \mathbb{R} | a < x < b\} \quad \text{(open interval)} \\
[a,b] \;&:= \{x \in \mathbb{R} | a \le x \le b\} \quad \text{(closed interval).}
\end{aligned}$$

$a$ and $b$ are the boundary points of these intervals.

**Definition 4.** *Let $V, \|\cdot\|$ be a normed vector space. Then, a set $\mathcal{O} \subset V$ is called an open set, if*

$$\forall x \in \mathcal{O} \quad \exists \varepsilon > 0 : \mathcal{U}_\varepsilon(x) \subset \mathcal{O}$$

*where*

$$\mathcal{U}_\varepsilon(x) := \{y \in V \mid \|x - y\| < \varepsilon\}.$$

**Example 5.**

- In $\mathbb{R}$: $\mathcal{U}_\varepsilon(x) = \,]\, x - \varepsilon, x + \varepsilon\,[$.
- In $\mathbb{R}^d$: $\mathcal{U}_\varepsilon(x)$ is a ball of radius $\varepsilon$, where we apply the norm

$$\|\vec{x}\|_2 := \sqrt{x_1^2 + \ldots + x_d^2}.$$

**Theorem 1.** *The collection of all open sets of a vector space is a topology.*

Proof: We have to prove that Definition 4 constructs a topology!

i) trivial

ii) Let $\mathcal{O}, P \in U$ be two open sets. Then let $x \in \mathcal{O} \cap P \Rightarrow x \in \mathcal{O}$ and $x \in P$. Then, we find $\varepsilon_1, \varepsilon_2 > O$ such that $U_{\varepsilon_1}(x) \subset O$ and $U_{\varepsilon_2}(x) \subset P$. Now, we get

$$\begin{aligned} \varepsilon := \min(\varepsilon_1, \varepsilon_2) &\Rightarrow U_\varepsilon(x) \subset \mathcal{O} \,\text{and}\, U_\varepsilon(x) \subset P \\ &\Rightarrow U_\varepsilon(x) \subset \mathcal{O} \cap P. \end{aligned}$$

This shows $O \cap P \in U$.

iii) Proof is left to the reader.

$\square$.

**Definition 5.** *Let $\mathcal{U}$ be a topology on the normed vector space $V$. A set $B \subset V$ is called closed set if and only if $V \backslash B \in \mathcal{U}$. Let $B \subset V$ be a subset of a normed vector space $V$. Then,*

$$\begin{array}{lcll} \dot{B} & = & \{x \in B \mid \exists \varepsilon > 0 : \quad U_\varepsilon(x) \subset B\} & \textit{is called interior of} \quad B \\ \overline{B} & = & \{x \in V \mid \forall \varepsilon > 0 : U_\varepsilon(x) \cap B \neq \emptyset\} & \textit{is called closure of} \quad B \\ \partial B & = & \{x \in V \mid \forall \varepsilon > 0 : U_\varepsilon(x) \cap B \neq \emptyset \wedge U_\varepsilon(x) \cap B^c \neq \emptyset\} & \textit{is called boundary of} \quad B \end{array}$$

*Here we abbreviate $B^c = V \backslash B$ .*

**Example 6.** $[a,b]$ is closed since $\mathbb{R}\backslash[a,b] = ]-\infty, a[ \cup ]b, \infty[$.
Interior of $[a,b]$ is $]a,b[$.
Interior of $]a,b[$ is $]\,a,b\,[$
Interior of $[a,b[$ is $]\,a,b\,[$

$$\begin{aligned}
\bigcup_{n\in\mathbb{N}}[-1,\frac{1}{n}] &= [-1,1] \\
\bigcap_{n\in\mathbb{N}}[\frac{1}{n},2] &= [1,2] \quad \text{closed} \\
\bigcup_{n\in\mathbb{N}}[\frac{1}{n},2] &= ]\,0,2\,] \leftarrow \begin{array}{l}\text{not open}\\ \text{not closed}\end{array}
\end{aligned}$$

**Definition 6.** *Let $V$ be a normed vector space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called* Cauchy sequence*, if $\forall\varepsilon > 0 \quad \exists k \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon \quad \forall n,m \geq k$.*

*The sequence $(x_m)_{m\in\mathbb{N}}$ converges to $x \in V$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. $V$ is called a Banach space, if every Cauchy sequence in $V$ converges.*

$\mathbb{R}^n$ is a Banach space for every norm on $\mathbb{R}^n$.

$$\begin{aligned}
\|(x_i)_{i=1,\dots,n}\|_\infty &= \max_{i=1}^{n} |x_i| \\
\|(x_i)_{i=1,\dots,n}\|_2 &= (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}
\end{aligned}$$

$\mathcal{C}([a,b])$ with norm $\|\cdot\|_{\mathcal{L}^2}$ is not a Banach space.

**Theorem 2.** *"Completeness": Let $V$ be a normed vector space, with norm $|||\cdot|||$. Then there exists a Banach space $B$ with norm $\|\cdot\|$ such that*

*a) $V \subset B$ is a subvector space and*

*b) $\|v\| = |||v||| \quad \forall v \in V$ .*

Proof (Idea): Define an equivalence relation on the set of Cauchy sequences. □

**Example 7** (Lebesque space)**.** The completeness of $R^2([a,b])$ leads to the Banach space $L^2([a,b])$. This space can be described as the set of Lebesque integrable functions. We integrate, add and multiply functions in this space according to functions in $R^2([a,b])$, but $L^2([a,b])$ just contains some "more" functions.

## 2.3 Equivalent Norms

**Definition 7.** *Let $\|\cdot\|, |||\cdot|||$ be norms of a vector space $V$. These norms are called equivalent, if there exist constants $k, c > 0$ such that:*

$$c\|v\| \leq |||v||| \leq k\|v\| \quad \forall v \in V.$$

**Example 8.** $\tilde{l}_p$ norms on $\mathbb{R}^n$.

$$\begin{aligned} \|(x_i)_{i=1,\dots,n}\|_{\tilde{l}^p} &= \left(\frac{1}{n}\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} \quad \text{for} \quad 1 \leq p < \infty \\ \|(x_i)_{i=1,\dots,n}\|_{\infty} &= \max_{i=1}^{n}|x_i| \end{aligned}$$

**Formula 1.** *Assume $p \geq 1$.*

*a)* $\|\vec{x}\|_{\tilde{l}^p} \leq \|\vec{x}\|_{\infty}$.

*b)* $\|\vec{x}\|_{\infty} \leq \|\vec{x}\|_{\tilde{l}^p} n^{\frac{1}{p}}$.

Proof: a)

$$\|\vec{x}\|_{\tilde{l}^p} = \left(\frac{1}{n}\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|\vec{x}\|_{\infty}^p\right)^{\frac{1}{p}} \leq n^{-\frac{1}{p}} n^{\frac{1}{p}} \|\vec{x}\|_{\infty} = \|\vec{x}\|_{\infty}.$$

b)

$$\|\vec{x}\|_{\infty} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} \leq \|\vec{x}\|_{\tilde{l}^p} n^{\frac{1}{p}}.$$

□

In Numerics: Use normalized norms $\|(1)_{i=1,\dots,n}\| = 1$ on $\mathbb{R}^n$.

**Example 9** (Finite difference discretization of Poisson's equation)**.** Let $u_n$ be the finite difference discretization of Poisson's equations with mesh size $h = \frac{1}{\sqrt{n}}$.

- Assume we prove (measure) $\|u_n - u\|_{\infty} \leq ch^2 = cn^{-1}$.
  Then we get $\|u_n - u\|_{\tilde{l}^p} \leq cn^{-1}$.
- Assume we prove $\|u_n - u\|_{\tilde{l}^2} \leq cn^{-1}$.
  Then we get $\|u_n - u\|_{\infty} \leq cn^{-1} \cdot n^{\frac{1}{2}} = c \cdot n^{-\frac{1}{2}}$.

**Theorem 3.** *All norms in $\mathbb{R}^n$ are equivalent.*

**Example 10.** $\mathbb{R}^\infty = \{(x_i)_{i\in\mathbb{N}} | x_i \in \mathbb{R}\}$.

$$\begin{array}{lcll} l_p & = & \{(x_i)_{i\in\mathbb{N}} | \sum_{i=1}^\infty |x_i|^p < \infty\}, & \text{where} \quad \|(x_i)_{i\in\mathbb{N}}\|_{l^p} := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}} \\ l_\infty & = & \{(x_i)_{i\in\mathbb{N}} | \sup_{i=1}^\infty |x_i| < \infty\}, & \text{where} \quad \|(x_i)_{i\in\mathbb{N}}\|_{l^\infty} := \sup_{i=1}^\infty |x_i|. \end{array}$$

**Formula 2.** $l_p \subset l_q$ *if* $p < q$.

Observe that the sequence $(1, 1, 1, ...) \in l_\infty$ but $(1, 1, 1, ...) \notin l_1$. This implies $l_1 \not\supset l_\infty$.

$\|\cdot\|_{l^1}$ is not equivalent to $\|\cdot\|_{l^\infty}$. This follows by the following counter-example:

$$\begin{array}{rcl} \vec{x}^1 & = & (1, 0, 0, 0 \dots \\ \vec{x}^2 & = & (1, 1, 0, 0 \dots \\ \vec{x}^3 & = & (1, 1, 1, 0, 0 \dots \\ \vec{x}^s & = & (x_i^s)_{i=1}^\infty \end{array}$$

$$x_i^s = \begin{cases} 1 & \text{if} \quad i \le s \\ 0 & \text{if} \quad i > s \end{cases}$$

$$\begin{array}{rcl} \|\vec{x}^s\|_{l^1} & = & \sum_{i=1}^\infty |x_i|^1 = s \\ \|\vec{x}^s\|_{l^\infty} & = & 1 \end{array}$$

## 2.4 Continuity, Linear Mappings

**Definition 8.** *Let $f : X \to Y$ be a mapping, where $X \subset V, Y \subset W$ and $V, W$ are normed vector spaces. $f$ is called continuous at $x_0 \in X$, if for every $\varepsilon > 0$ there exists a $\delta > 0$:*

$$\forall x : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon.$$

**Lemma 2.** *$f : X \to Y$ is continuous in $x$. $\iff$ Let $(x_i)$ be a sequence such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$.*

**Lemma 3.** *: The composition, sum and product of continuous functions is continuous.*

**Definition 9.** *A mapping between two vector spaces $f : V \to W$ is called linear, if:*

$$\begin{array}{rll} i) & f(x + y) = f(x) + f(y) & \forall x, y \in V, \\ ii) & f(\lambda x) = \lambda f(x) & \forall x \in V,\ \lambda \in \mathbb{K}. \end{array}$$

**Theorem 4.** *Let $T : X \to Y$ be a linear mapping between normed vector spaces. Then, the following statements are equivalent:*

*a) $T$ is continuous at a fixed point $x_0$.*

*b) $T$ is continuous in $x = 0$.*

*c) $\|T\|_{X\to Y} := \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|T(x)\|_Y}{\|x\|_X}$ is bounded.*

Proof

a) $\Rightarrow$ b) trivial.

b) $\Rightarrow$ c) For $\varepsilon = 1$ let $\delta > 0$ such that $\forall \tilde{x} : \|\tilde{x} - 0\|_X \leq \delta \Rightarrow \|T(\tilde{x}) - T(0)\|_Y \leq 1$ .

Now let $x \neq 0$. Then, define $\tilde{x} = \frac{x}{\|x\|_X}\delta$. Observe $\|\tilde{x}\|_X = \delta$. Therefore, we get

$$1 \geq \|T(\tilde{x})\|_Y = \delta\frac{\|T(x)\|_Y}{\|x\|_X} \Rightarrow \frac{\|T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta}.$$

This implies

$$\|T\|_{X\to Y} \leq \frac{1}{\delta}.$$

c) $\Rightarrow$ a) Let $\varepsilon > 0$ and choose

$$\delta = \frac{\varepsilon}{\|T\|_{X\to Y}}.$$

Assume that $\|x - x_0\|_X < \delta$. Then, we get

$$\begin{aligned}\|T(x) - T(x_0)\|_Y &= \|T(x - x_o)\|_Y \leq \|T\|_{X\to Y}\|x - x_0\|_X \\ &\leq \|T\|_{X\to Y}\delta \leq \varepsilon.\end{aligned}$$

□

**Lemma 4.** *Let $T : X \to Y$, $S : Y \to Z$ be continuous linear mappings between normed vector spaces. Then we get:*

*a) $\|T\|_{X\to Y} = \sup_{\substack{x\in X\\ \|x\|=1}} \|T(x)\|_Y$. ($\|T\|$ is called operator norm.)*

*b)* $\|ST\|_{X\to Z} \leq \|S\|_{Y\to Z}\|T\|_{X\to Y}$.

*c)* $\|T(x)\| \leq \|T\|\|x\| \quad \forall x \in X$.

**Definition 10.** *Let $X, Y >$ be normed vector spaces.*
*Then, let $\mathcal{L}(X,Y) = \{f : X \to Y \mid f \text{is linear and continuous}\}$.*
*In case of $X = Y$ we write $\mathcal{L}(X)$.*

**Theorem 5.** *$(\mathcal{L}(X,Y), \|\cdot\|_{X\to Y})$ is a normed space. If $Y$ is a Banach space, then, $\mathcal{L}(X,Y)$ is a Banach space.*

*Proof:*
Let us prove: If $(T_k)$ is a Cauchy sequence in $\mathcal{L}(X,Y)$, then $(T_k)$ converges in $\mathcal{L}(X,Y)$.
Let us define: $T(x) = \lim_{k\to\infty} T_k(x)$. This limit exists since, the inequality

$$|T_k(x) - T_l(x)| = |(T_k - T_l)(x)| \leq \|T_k - T_l\|\|x\|$$

shows that $(T_k(x))_k$ is a Cauchy sequence for every $x$.

1) Let us first prove that the mapping

$$\begin{aligned} T: \quad & X \to Y \\ & x \mapsto y \quad = \lim_{k\to\infty} T_k(x) \end{aligned}$$

is linear:

$$\begin{aligned} T(x+z) &= \lim_{k\to\infty} T_k(x+z) = \lim_{k\to\infty} T_k(x) + T_k(z) \\ &= \lim_{k\to\infty} T_k(x) + \lim_{k\to\infty} T_k(z) = T(x) + T(z). \end{aligned}$$

2) T is continuous: Observe that

$$\|T(x)\| = \lim_{k\to\infty} \|T_k(x)\| \leq \sup_k \|T_k\|\|x\|, \tag{1}$$

since $\|T_k(x)\| \leq \|T_k\|\|x\| \leq \sup_k \|T_k\| \cdot \|x\|$. Since $(T_k)$ is a Cauchy sequence, there is a $\tilde{k}$ such that $\|T_k - T_l\| < 1 \ \forall k, l \geq \tilde{k}$. Therefore, we get:

$$\|T_k\| \leq \|T_k - T_{\tilde{k}} + T_{\tilde{k}}\| \leq \|T_k - T_{\tilde{k}}\| + \|T_{\tilde{k}}\| \leq 1 + \|T_{\tilde{k}}\| \quad \forall k \geq \tilde{k}.$$

This implies:

$$\|T_k\| \leq \max_{l=1}^{\tilde{k}} \|T_l\| + 1 \quad \forall k \geq 0. \tag{2}$$

Theorem 4 and Equation (1) and (2) imply that $T$ is continuous.

Now we prove: $\lim_{k\to\infty} \|T_k - T\| = 0$. Let $\varepsilon > 0$. Since $(T_k)$ is a Cauchy sequence there exists a $\tilde{k}$ such that:

$$\|T_k - T_l\| \leq \varepsilon \quad \forall k, l \geq \tilde{k}$$

$$\begin{array}{rlcl} \Rightarrow & \|T_k(x) - T_l(x)\| \leq \|(T_k - T_l)(x)\| & \leq & \|T_k - T_l\|\|x\| \leq \varepsilon\|x\| \\ & \|T_k(x) - T(x)\| = \lim_{l\to\infty} \|T_k(x) - T_l(x)\| & \leq & \varepsilon\|x\| \\ & & \Rightarrow & \|T_k - T\| \leq \varepsilon. \end{array}$$

This implies $\lim_{k\to\infty} \|T_k - T\| = 0$.

□

**Definition 11.** *Let $X$ be normed space. Then $X' = \mathcal{L}(X, \mathbb{K})$ is called the dual space. $f \in X'$ is called a linear functional.*

# 3 Hilbert Spaces

**Definition 12.** *Let $H$ be a vector space over $\mathbb{K} = \mathbb{C}$ on $\mathbb{R}$. A mapping $< \cdot, \cdot >: H \times H \to \mathbb{K}$ is called scalar product, if*

*a)* $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

*b)* $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

*c)* $\langle x, x\rangle > 0$ *for* $x \neq 0$.

Remark:

i) $\langle x, x\rangle \in \mathbb{R}$.

ii) $\langle , \rangle$ is a sesquilinear form $\begin{array}{lcll} x & \to & \langle x, y\rangle & \text{linear} \\ y & \to & \langle x, y\rangle & \text{anti-linear} \end{array}$

Proof:

$$\begin{array}{ccc} \bar{\alpha}\langle z, x\rangle + \bar{\beta}\langle z, y\rangle & = & \overline{\alpha\langle x, z\rangle + \beta\langle y, z\rangle} \\ & & \| \\ \langle z, \alpha x + \beta y\rangle & = & \overline{\langle \alpha x + \beta y, z\rangle} \end{array}$$

**Theorem 6.** *: Let $H$ be a vector space with scalar product. Then, $\|x\|_H = \sqrt{\langle x, x\rangle}$ is a norm. Furthermore, the following Cauchy-Schwarz inequality holds $|\langle x, y\rangle| \leq \|x\|_H\|y\|_H$.*

Proof: Let $x, y \in H, \alpha \in \mathbb{K} \quad x \neq 0$:

$$0 \leq \langle \alpha x + y, \alpha x + y \rangle = |\alpha|^2 \|x\|_H^2 + 2 \ \mathrm{Re}(\alpha \langle x, y \rangle) + \|y\|_H^2.$$

Choose: $\alpha = -\overline{\langle x, y \rangle}/\|x\|_H^2$. Then, we get:

$$0 \leq \|\alpha x + y\|_H^2 = \|y\|_H^2 - \frac{|\langle x, y \rangle|^2}{\|x\|_H^2} \Rightarrow \quad \text{Cauchy-Schwarz inequality}$$

Obviously, $\|\lambda x\|_H = |\lambda| \|x\|_H$. To prove that $\| \cdot \|$ is norm, it is enough to show that the triangle inequality holds:

$$\begin{aligned} \|x + y\|_H^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \\ &= \|x\|_H^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|_H^2 \\ &= \|x\|_H^2 + 2\mathrm{Re}\langle x, y \rangle + \|y\|_H^2 \\ &\leq \|x\|_H^2 + 2\|x\|_H \|y\|_H + \|y\|_H^2 \\ &= (\|x\|_H + \|y\|_H)^2 \end{aligned}$$

□

Remark: By the Cauchy-Schwarz inequality, we can show the continuity $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$, where the norm $\|(v, w)\|_{H \times H} = \max(\|v\|_H, \|w\|_H)$ is used.

**Definition 13.** *A vector space $H$ with scalar product is called pre-Hilbert space. $H$ is called Hilbert space, if $H$ is Banach space with respect to $\|u\| = \sqrt{\langle u, u \rangle}$.*

**Example 11.**

- $\mathbb{C}^n$ is a Hilbert space with scalar product:

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \bar{y}_i.$$

- $L^2(]\,0, 1\,[)$ is Hilbert space with scalar product $\langle u, v \rangle = \int_0^1 u \cdot v dx$.
- $L^2(\Omega)$ is Hilbert space with scalar product $\langle u, v \rangle = \int_\Omega u \cdot v dy$, where $\Omega \subset \mathbb{R}^d$ is an open subset.

- $\mathcal{C}^1([0,1]) := \{f \in \mathcal{C}([0,1]) | f' \in \mathcal{C}([0,1])\}$ is pre-Hilbert space with scalar product $\langle f, g\rangle_{H^1} := \int_o^1 f' \cdot g' + f \cdot g \quad dx$.

  The completeness of this pre-Hilbert space leads to the Hilbert space $H^1(]\,0,1\,[)$.

<u>Problem:</u> Find $\alpha \in \mathbb{R}$ such that

$$x^\alpha \in H^1(]\,0,1\,[).$$

**Definition 14.** *Let $V, W$ be subspaces of the real pre-Hilbert space $H$. Then,*

$$\gamma = \sup_{\substack{x \in V, y \in W \\ x,y \neq 0}} \frac{|\langle x, y\rangle|}{\|x\|\|y\|}$$

*is called constant in the strengthened Cauchy-Schwarz inequality between $V$ and $W$.* $\arccos(\gamma) =: \angle(V, W)$ *is called the angle between $V, W$.*

**Lemma 5.** *Let $K \geq 1$ be a constant. Then the following equivalence holds:*

$$\begin{aligned} \|v\|^2 + \|w\|^2 &\leq K\|v + w\|^2 \qquad \forall v \in V, w \in W \\ &\Updownarrow \\ \gamma(V, W) &\leq \frac{K-1}{K} \end{aligned}$$

<u>Proof:</u>

$\Uparrow$: $- < v, w > \leq \dfrac{K-1}{K}\|v\|\|w\|$

$$\begin{aligned} 0 \leq (\|v\| - \|w\|)^2 \;&\Rightarrow\; 2\|v\|\|w\| \leq \|v\|^2 + \|w\|^2 \\ &\Rightarrow\; -2 < v, w > K \leq (K-1)(\|v\| + \|w\|)^2 \\ &\Rightarrow\; \|v\|^2 + \|w\|^2 \leq K(\|v + w\|^2. \end{aligned}$$

$\Downarrow$ Choose $v \in V, w \in W$ such that $\|v\| = \|w\| = 1$. Then, we get

$$\begin{aligned} \|v\|^2 + \|w\|^2 &\leq K\|v + (-w)\|^2 \\ &\Downarrow \\ 2 < v, w > K &\leq (K-1)\|v\|^2 + (K-1)\|w\|^2 \\ &\Downarrow \\ < v, w > K &\leq (K-1)\|v\|\|w\|. \end{aligned}$$

□

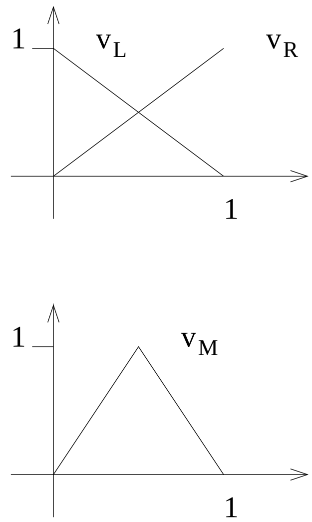



Figure 1: Hierarchical Basis.

**Example 12.** *See Figure 1.*

$$\int_0^1 uvdx =< u, v >_{L^2}, \qquad W = \{v_M\rho | \rho \in \mathbb{R}\},$$
$$V = \{v_L\beta + \alpha v_R | \alpha, \beta \in \mathbb{R}\}$$
$$\gamma(W, V) \approx 0.8603.$$

**Definition 15.** *Let $H$ be a Hilbert space.*

*a) $d = \dim(H) < \infty$. Then $(b_n)_{n=1,\ldots,d}$ is a Hilbert space basis, if*

$$\langle b_n, b_m \rangle = \delta_{n,m}.$$

*b) $dim(H) = \infty$. Then $(b_n)_{n\in\mathbb{N}}$ is a Hilbert space basis, if*

- *for every $x \in H$ there exists a unique sequence $(\lambda_i)_{i\in\mathbb{N}}$, $\lambda_i \in \mathbb{K}$ such that*

$$x = \sum_{i=1}^{\infty} \lambda_i b_i$$

- *and*

$$\langle b_n, b_m \rangle = \delta_{n,m}.$$

Problem: Assume that $(b_i)_{i\in\mathbb{N}}$ is Hilbert space basis. Then, for every $x \in H$ exists $(\lambda_i)_{i\in\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \lambda_i b_i$. How can we compute $\lambda_i$?

**Formula 3.** *Let $(b_i)_{i\in\mathbb{N}}$ be a Hilbert space basis. Let $x = \sum_{i=1}^{\infty} \lambda_i b_i$. Then, $\lambda_i = \langle x, b_i\rangle$ for every $i \in \mathbb{N}$.*

*Furthermore, the following formula holds:*

$$\|u\| = \sqrt{\sum_{i=1}^{\infty} |\lambda_i|^2}.$$

Proof:

$$\langle x, b_i\rangle = \langle \sum_{i=1}^{\infty} \lambda_j b_j, b_i\rangle = \sum_{j=1}^{\infty} \lambda_j \langle b_j, b_i\rangle = \lambda_i$$

and

$$\begin{aligned}
\|x\|^2 &= \langle x, x\rangle \\
&= \langle \sum_{j=1}^{\infty} \lambda_j b_j, \sum_{i=1}^{\infty} \lambda_i b_i\rangle \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \lambda_j \lambda_i \langle b_j, b_i\rangle = \sum_{i=1}^{\infty} |\lambda_i|^2.
\end{aligned}$$

□

Remark: The functions $\sin(nx), \cos(nx)$ or $\exp(inx)$ lead to Hilbert space basis. Then, Formula 3 is the formula of Fourier transformation.

Observe that a Hilbert space basis is not a basis of $H$ in the classical sense.

**Example 13.**

a) A Hilbert space basis in $\mathbb{R}^n$ are the unit vectors:

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \quad i\text{-th component in } \mathbb{R}^n$$

b) Consider the Hilbert space $l_2 = \{(x_i)_{i\in\mathbb{N}} | \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with scalar product $\langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \rangle = \sum_{i=1}^{\infty} x_i y_i$. Then

$$e^j = (e_i^j)_{i\in\mathbb{N}}, \quad e_i^j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{else} \end{cases}$$

is a Hilbert space basis.

Proof: Let $\vec{x} = (x_i)_{i\in\mathbb{N}}$. We want to prove: $\vec{x} = \sum_{j=1}^{\infty} x_j e^j$. This follows by

$$\|\vec{x} - \sum_{j=1}^{n} x_j e^j\|_{l^2} = \left( \sum_{i=n+1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \longrightarrow 0 \quad \text{for } n \to \infty.$$

$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \end{pmatrix} \right\|_{l^2} = \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_{n+1} \\ \vdots \end{pmatrix} \right\|_{l^2}$$

c) Consider the space (see Example 7)

$$H = L^2_{\mathbb{R}}(]\,0, 2\pi\,[)$$

with scalar product

$$\langle u, w \rangle_{L^2} = \int_0^{2\pi} u(x) w(x)\, dx.$$

A Hilbert space basis of this space $L^2_{\mathbb{R}}(]\,0, 2\pi\,[)$ is

$$S = \left\{ \frac{1}{\sqrt{2\pi}} \underline{\underline{1}} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \cos(mx) \big| m \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin(mx) \big| m \in \mathbb{N} \right\}$$

where

$$\begin{array}{rcl} \underline{\underline{1}} : ]\,0, 2\pi\,[ & \to & \mathbb{R} \\ x & \to & 1 \end{array}.$$

Observe that

$$\left\| \frac{1}{\sqrt{2\pi}} \underline{\underline{1}} \right\|_{L^2} = \sqrt{\int_0^{2\pi} \left( \frac{1}{\sqrt{2\pi}} \right)^2 dx} = 1.$$

This basis leads to the real Fourier series decomposition. To prove that every function can be approximated by the above Hilbert space basis, one studies the complex Fourier series decomposition.

Instead of $H = L^2_{\mathbb{R}}(]\,0, 2\pi\,[)$ one can also define the space

$$H = L^2_{\mathbb{R}}(]-\pi, \pi\,[)$$

d) The space

$$H = L^2_{\mathbb{R}}(]\,0, \pi\,[)$$

with scalar product

$$\langle u, w\rangle_{L^2} = \int_0^\pi u(x)w(x)\,dx$$

is a Hilbert space basis, too. Now, a Hilbert space basis is

$$\left\{ \frac{2}{\sqrt{\pi}} \sin(nx) \middle| n \in \mathbb{N} \right\}.$$

Observe, that the cosinus functions are orthogonal to these functions on $[-\pi, \pi]$. To prove that these functions are a Hilbert space basis, apply the mapping

$$L^2_{\mathbb{R}}(]\,0, \pi\,[) \ni f \mapsto \left( x \mapsto \begin{array}{cl} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x \leq 0. \end{array} \right) \in L^2_{\mathbb{R}}(]-\pi, \pi\,[)$$

The Fourier series of a function obtained by this mapping does not contain cosinus terms.

e) Consider $H = L^2_{\mathbb{C}}(]\,0, 2\pi\,[)$ with scalar product

$$\langle u, w\rangle_{L^2} = \int_0^{2\pi} u(x)\overline{w(x)}\,dx.$$

Then $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n\in\mathbb{Z}}$ is a Hilbert space basis. ( complex Fourier series decomposition).

<u>Proof</u> of orthogonality:

$$\begin{aligned} \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{inx} \frac{1}{\sqrt{2\pi}} \overline{e^{imx}} dx &= \int_0^{2\pi} \frac{1}{2\pi} e^{i(n-m)x} dx = \\ &= \left\{ \begin{array}{cl} 1 & \text{if} \quad n = m \\ \frac{1}{i(n-m)} \frac{1}{2\pi} e^{i(n-m)x}\Big|_0^{2\pi} = 0 & \text{else} \end{array} \right. . \end{aligned}$$

The difficulty is to show that every function can be approximated by the Hilbert space basis $\left(\frac{1}{\sqrt{2\pi}}e^{inx}\right)_{n\in\mathbb{Z}}$. To prove this property, observe that

$$\sum_{n\in\mathbb{Z}} e^{inx}\lambda_n = \sum_{n\in\mathbb{Z}}(e^{ix})^n\lambda_n.$$

Then, one applies the approximation property of polynomials.

d) $\mathbb{C}^n$ is a Hilbert space with scalar product

$$\langle (x_j),(y_j)\rangle = \sum_{j=1}^{n} x_j\bar{y}_j.$$

Then, $(b_p)_{p=1,\dots,n}$, where

$$b_p = \frac{1}{\sqrt{n}}(e^{i\frac{2\pi}{n}pj})_{j=1,\dots,n}$$

is Hilbert space basis of $\mathbb{C}^n$.

**Theorem 7.** *(German: Hauptachsentransformation, English: principal axis theorem). Let $A$ be a symmetric matrix over $\mathbb{R}$. Then there is an orthogonal matrix $B$ such that $B^TAB = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$, where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $A$. The colums of $B$ are the corresponding eigenvectors $b_1,\dots,b_n$. Orthogonality of $B$ means $B^TB = E$. This is equivalent to*

$$\langle b_i, b_j\rangle = \delta_{ij}.$$

**Definition 16.** *Let $H$ be a Hilbert space and $f : H \to H$, continuous and linear. $f$ is called selfadjoint, if*

$$< f(x), y >=< x, f(y) > \quad \forall x,y\in H.$$

Using this definition, Theorem 7 can be described as follows.

**Theorem 8.** *Let $f : H \to H$ be linear and selfadjoint and $H = \mathbb{R}^n$. Then, there exists an orthonormal basis of eigenvectors $(e^j)_{j=1,\dots,n}$ in $H$. The matrix corresponding to $f : \mathbb{R}^n \to \mathbb{R}^n$ with respect to $(e^j)_{j=1,\dots,n}$ is an $n\times n$ diagonal matrix. This means*

$$f(e^j) = \lambda_j e^j, \qquad \lambda_j\in\mathbb{R}.$$

**Example 14.** Let $\Omega_h := \{ih | i = 0, ..., n-1\}$, $h = \frac{1}{n}$ be a discretization grid of $[0, 1[$. Let us extend $\Omega_h$ periodically $\Omega_h^\infty = \mathbb{Z}h$.

The space of 1-periodic functions is defined by

$$\mathcal{F}_{1,per}(\Omega_h^\infty) = \{f : \Omega_h^\infty \to \mathbb{C} \mid f(p) = f(p+1) \quad \forall p \in \Omega_h^\infty\}.$$

This space is isomorph (has the same structure as) to $\mathbb{C}^n$. An isomorphism is given by

$$\begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} \to (ih \longmapsto a_{i \ mod \ n})$$

Consider the finite difference operators

$$\begin{aligned}
\delta_h^1(u)(p) &= \frac{u(p+h) - u(p-h)}{2h}, \\
\delta_h^2(u)(p) &= \frac{u(p+h) - 2u(p) + u(p-h)}{h^2}, \\
\delta_h^{1,l}(u)(p) &= \frac{u(p+h) - u(p)}{h}.
\end{aligned}$$

$\mathcal{F}_{1,per}(\Omega_h^\infty)$ is Hilbert space with scalar product:

$$< u, v >= \sum_{i=0}^{n-1} u(ih)\overline{v(ih)}.$$

<u>$\delta_h^2$ is selfadjoint in $\mathcal{F}_{1,per}(\Omega_h^\infty)$</u>

$$\begin{aligned}
< \delta_n^2(u), v > &= \sum_{i=0}^{n-1} \left( \frac{u(ih+h) - 2u(ih) + u(ih-h)}{h^2} \right) \quad \overline{v(ih)} \\
&= \frac{1}{h^2} \Bigg( \quad \sum_{i=0}^{n-1} u((i-1)h + h)\overline{v((i-1)h)} \\
&\quad -2 \sum_{i=0}^{n-1} u(ih)\overline{v(ih)} + \sum_{i=0}^{n-1} u((i+1)h - h)\overline{v((i+1)h)} \Bigg) \\
&= \sum_{i=0}^{n-1} u(ih) \left( \frac{\overline{v(ih-h)} - 2\overline{v(ih)} + \overline{v(ih+h)}}{h^2} \right) =< u, \delta_n^2(v) >
\end{aligned}$$

The eigenvectors of $\delta_n^2$ are:

$b_q := \frac{1}{\sqrt{n}}(e^{i2\pi qx})_{x\in\Omega_h^\infty}$, $q = 0, ..., n-1$.

The corresponding eigenvalue is:

$$\begin{aligned}
&\frac{1}{h^2}(e^{i2\pi qh} + e^{-i2\pi qh} - 2) = \\
&= \frac{1}{h^2}(\cos(2\pi qh) + i\sin(2\pi qh) - 2 + \cos(-2\pi qh) + i\sin(-2\pi qh)) \\
&= \frac{1}{h^2}(2\cos(2\pi qh) - 2) \\
&= \frac{2}{h^2}(\cos(2\pi qh) - 1).
\end{aligned}$$

**Theorem 9** (Riesz Representation Theorem)**.** *Let $H$ be a Hilber space. For every $f \in H'$ exists a $y \in H$, such that*

$$< x.y >= f(x) \quad \forall x \in H.$$

*Furthermore, $\|y\|_H = \|f\|_{H^1}$. $y$ is the unique solution of the following minimization problem*
*Find $x \in H$, such that $F(x) :=< x, x > -2Ref(x)$ minimal.*

Idea of proof:

$$F(y) =< y, y > -2Ref(y) \geq \|y\|_H^2 - 2\|f\|_{H^1}\|y\|_H \geq -\|f\|_{H^1}^2.$$

Therefore, there is a sequence $(y_k)_{k\in\mathbf{N}}$ such that $\lim_{k\to\infty} F(y_k) = \inf_{y\in H} F(y)$.

By Parallelogram equation one can prove that $(y_k)$ is Cauchy sequence. Let

$$y := \lim_{k\to\infty}(y_k).$$

Furthermore

$$\begin{aligned}
0 = \frac{d}{dt}F(y+tx)|_{t=0} &= \\
&= \frac{d}{dt}\big(t^2 < x, x > +2tRe < x, y > + < y, y > -2tRef(x) - 2Ref(y)\big)|_{t=0} \\
\Rightarrow 0 &= 2Re(< x, y > -f(x)) \\
&\Rightarrow Re(< x, y >) = Re(f(x)).
\end{aligned}$$

Analogously, one can prove

$$Re(i < x, y >) = Re(if(x)).$$

This implies

$$< x, y >= f(x) \quad \forall x.$$

Proof of isometric mapping:

$$\begin{aligned}
\|f\|_{H'} &:= \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \neq 0} \frac{|< x, y >|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|x\| \cdot \|y\|}{\|x\|} \leq \|y\|, \\
\Rightarrow \|f\|_{H'} &\leq \|y\|_H, \\
\frac{|f(y)|}{\|y\|} &= \frac{< y, y >}{\|y\|} = \frac{\|y\|^2}{\|y\|}, \\
\Rightarrow \|f\|_{H'} &\geq \|y\|.
\end{aligned}$$

This shows $\|f\|_{H'} = \|y\|$.

□

# 4 Sobolev-Spaces

## 4.1 Basic Definitions

**Definition 17.** *Let $K \subset \mathbb{R}^d$. $K$ is called compact, if $K$ is closed and bounded.*

**Example 15.** *A square domain $[2, 4]^2$ in $\mathbb{R}^2$ is compact. A set of 12 points is compact.*

**Definition 18.** *Let $\Omega \subset \mathbb{R}^d$ be open. The support of a function $f : \Omega \to \mathbb{R}$ is defined by*

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}}$$

*Now, let us define the vector space*

$$\begin{aligned}
\mathcal{C}_0^\infty(\Omega) &= \{f : \Omega \to \mathbb{K} \mid \text{supp}(f) \subset \Omega \\
&\quad \text{is compact and } f \text{ is arbitrary often differentiable}\}
\end{aligned}$$

**Example 16** (Mollifier Function)**.**

$$\mathcal{M}(x) = \begin{cases} K \exp \frac{1}{|x|^2 - 1} & \text{for} \quad |x| < 1 \\ 0 \quad \text{else.} \end{cases}$$

*$K$ is chosen such that $\int_{-\infty}^{\infty} \mathcal{M}(x) dx = 1$.*

**Theorem 10.** *$\mathbb{C}_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. This means that for every $u \in L^2(\Omega)$ exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $L^2(\Omega)$.*

This theorem can be proved by the convolution of $f$ and the Mollifier function. This leads to an arbitrary smooth function. Later, we will show that the Mollifier function converges to the delta distribution. The convolution with this delta distribution is the identity.

A multiindex is $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d \qquad |\alpha| := \alpha_1 + ... + \alpha_d$.

$$D^\alpha(\rho) = \frac{d^{|\alpha|}\rho}{dx_1^{\alpha_1} \cdot ... \cdot dx_d^{\alpha_d}}.$$

**Example 17.** *$D^\alpha \rho \in C_0^\infty(\Omega)$ for every $\rho \in C_0^\infty(\Omega)$.*

**Definition 19.** *The weak derivative of $u \in L^2(\Omega)$ with respect to $\alpha$ is the function $g \in L^2(\Omega)$, if the following holds:*

$$\int_\Omega u D^\alpha \rho = (-1)^{|\alpha|} \int_\Omega g\rho \quad \forall \rho \in C_0^\infty(\Omega).$$

*Let us abbreviate $D^\alpha u := g$.*

The classical derivative of a function coincides with the weak derivative for differentiable functions. To show this, let $u \in \mathcal{C}^1(\mathbb{R})$, $\varphi \in C_0^\infty(\mathbb{R})$ and observe that

$$\int_{-\infty}^{\infty} u \frac{d\varphi}{dx} dx = [u \cdot \varphi]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{du}{dx} \varphi dx = - \int_{-\infty}^{\infty} \frac{du}{dx} \varphi dx.$$

Let us prove that the weak derivative is unique. To this end let $g_1, g_2 \in L^2(\Omega)$ be the weak derivative of $u$. Then, we get

$$0 = \int_2 (g_1 - g_2)\varphi \quad \forall \varphi \in \mathbb{C}_0^\infty(\Omega).$$

By Theorem 10 $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. Thus, there is a sequence $\varphi_n \in \mathbb{C}_0^\infty(\Omega)$ such that $\lim_{n\to\infty} \varphi_n = \overline{g_1 - g_2}$.

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_\Omega (g_1 - g_2)\varphi_n = \int_\Omega (g_1 - g_2)\overline{(g_1 - g_2)} \\ &= \int_\Omega |g_1 - g_2|^2 = \|g_1 - g_2\|_{L^2}^2. \end{aligned}$$

This implies $g_1 = g_2$ in $L^2(\Omega)$. Here, observe that $L^2(\Omega)$ is a quotient space of $\mathcal{L}^2(\Omega)$.

**Example 18.** *Let $u(x) := |x|$. Then, $\frac{du}{dx} = \operatorname{sgn}(x)$.*

*Proof:*

$$\begin{aligned}
\int_{-\infty}^{\infty} u\frac{d\varphi}{dx} &= \int_{-\infty}^{\infty} |x|\frac{d\varphi}{dx} \\
&= -\int_{-\infty}^{0} \frac{d(-x)}{dx}\varphi dx + [(-x)\varphi]_{-\infty}^{0} - \int_{0}^{\infty} \frac{d(x)}{dx}\varphi dx + [x\varphi]_{0}^{\infty} \\
&= -\int_{\infty}^{\infty} \operatorname{sgn}(x)\varphi(x).
\end{aligned}$$

**Example 19.** *Let $u = \operatorname{sgn}(x)$. The weak derivative of $u$ does not exist.*

*Proof:*

$$\begin{aligned}
\int_{-\infty}^{\infty} u\frac{d\varphi}{dx} = \int_{-\infty}^{0} (-1)\frac{d\varphi}{dx}dx + \int_{0}^{\infty} 1\frac{d\varphi}{dx}dx \\
= [(-1)\varphi]_{-\infty}^{0} + [1\varphi]_{0}^{\infty} = -2\varphi(0)
\end{aligned}$$

*There is no function $g \in L^2(\mathbb{R})$ such that*

$$\int_{-\infty}^{\infty} g\varphi \; dx = \varphi(0) \quad \forall \varphi \in \mathbb{C}_0^{\infty}(\mathbb{R}).$$

*Therefore, the weak derivative of $u$ does not exist.*

**Definition 20** (Sobolev Space)**.** *Let $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ open. The Sobolev space of order $m$ with respect to $p = 2$ is defined by*

$$\begin{aligned}
W^m(\Omega) = \{f \in L^2(\Omega) \quad | \quad & f \text{ is } m\text{-times weak differentiable and} \\
& D^\alpha f \in L^2(\Omega),\, \forall |\alpha| \leq M\}.
\end{aligned}$$

*The norm in $W^m(\Omega)$ is*

$$\|u\|_{W^m(\Omega)} := (\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2)^{1/2}.$$

*The Sobolev space of order $m$ with respect to $p = \infty$ is*

$$W_{p=\infty(\Omega)}^m := \{f \in L^\infty(\Omega) | D^\alpha f \in L^\infty(\Omega), \quad \forall |\alpha| \leq m\}.$$

*The norm in $W_{p=\infty}^m(\Omega)$ is*

$$\|u\|_{W_{p=\infty}^m(\Omega)} := \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

**Example 20.** $W^1(\mathbb{R}^2)$*:*

$$\|u\|_{W^1} = \sqrt{\|u\|_{L^2}^2 + \|\frac{du}{dx}\|_{L^2}^2 + \|\frac{du}{dy}\|_{L^2}^2}.$$

**Definition 21.** *Let us define*

$$\mathring{W}^m(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{W^m(\Omega)}.$$

One can prove:

$$\begin{array}{rcl} W^0(\Omega) & = & L^2(\Omega), \\ \mathring{W}^1(\Omega) & \neq & W^1(\Omega). \end{array}$$

One can prove (see Trace Theorem) that $\mathring{W}^1(\Omega)$ consists of functions which are 0 at the boundary.

$W^m_{p=\infty}(\Omega)$ is a Banach space. $W^m(\Omega)$ ist Hilbert space with scalar product

$$(u,v) \to \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u \overline{D^\alpha v}$$

and norm $\|u\|_{W^m(\Omega)}$.

**Theorem 11** (First Poincare's Inequality)**.** *Assume that* $\Omega$ *is bounded. Then,*

$$|u|_{W^1} := \sqrt{\int_\Omega (Du)^2} = \sqrt{\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^2}$$

*is a norm on* $\mathring{W}^1(\Omega)$ *which is equivalent to* $\|u\|_{W^1}$*.*

Proof:
Observe that

$$\|w\|^2_{W^1(\Omega)} = |w|^2_{W^1(\Omega)} + \|w\|^2_{L^2(\Omega)} \geq |w|^2_{W^1(\Omega)}.$$

This is the first inequality, which hast to be proved. The second is

$$\|w\|^2_{W^1(\Omega)} \leq |w|^2_{W^1(\Omega)}.$$

Observe, that it is enough to show that there is a constant $K$ such that

$$\|w\|^2_{L^2(\Omega)} \leq K|w|^2_{W^1(\Omega)} \tag{3}$$

for every $w \in \overset{\circ}{W}{}^1(\Omega)$. It is enough to thow (3) for functions $\varphi \in \mathcal{C}_0^\infty(\Omega)$, since $\mathcal{C}_0^\infty(\Omega)$ is dense in $\overset{\circ}{W}{}^1(\Omega)$. Here, let us prove (3) for a one dimensional interval $\Omega = ]a, b[ \subset \mathbb{R}$. Then,

$$\int_{-\infty}^{x} \varphi' dt = \varphi(x) \Rightarrow |\varphi(x)| \leq \int_\Omega |\varphi'| dt.$$

This implies

$$\begin{aligned} \int_\Omega |\varphi(x)|^2 dx &\leq \int_\Omega (\int_\Omega |\varphi'| dt)^2 dx = \left(\int_\Omega |\varphi'| dt\right)^2 \cdot vol(\Omega) \\ &\leq \int_\Omega |\varphi'|^2 \; dt \cdot (vol(\Omega))^2 . \end{aligned}$$

The last inequality follows by

$$\int_\Omega |\varphi'| \cdot 1 dt \leq \quad \|\varphi'\|_{L^2(\Omega')} \cdot \quad \|1\|_{L^2(\Omega)}$$

□

## 4.2 Poisson's Problem

Poisson's problem can be described as follows:

**Poisson's Problem with homogenuous Dirichlet Boundary Conditions**

Let $f \in L^2(\Omega)$. Find $u \in \overset{\circ}{W}{}^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v d(x, y) = \int_\Omega f v d(x, y) \qquad \forall v \in \overset{\circ}{W}{}^1(\Omega)$$

An important application of the theory of Sobolov is that Poisson's problem has a unique solution.

Proof:

- By Poincare's inequality, $\mathring{W}^1(\Omega)$ is Hilbert space with scalar product:

$$(u, v) \to \int_\Omega \nabla u \nabla v d(x, y).$$

- $v \to \int_\Omega f v d(x, y)$ is a mapping contained in $(\mathring{W}^1(\Omega))'$ (dual space). This follows by Theorem 4 and

$$\left| \int_\Omega f \cdot v d(x, y) \right| \leq \|f\|_{L^2} \cdot \|v\|_{W^1} \leq \|f\|_{L^2} \cdot C \cdot |v|_{W^1},$$

  where the constant $C$ is obtained by Poincare's inequality.

- By Riesz Representation Theorem there is a unique solution of Poisson's equation.

This concept of proofing existence and uniquness of a partial differential equation can by extended to a large number of partial differential equations, which can be described in a weak form. This means that a funtional in the dual space and a bilinear form is given. As a second example consider the bilinear form

$$a(u, u) = \int_\Omega \nabla u^T A \nabla u d(x, y),$$

where is $A \in (L^\infty(\Omega))^{2\times 2}$ and $A(x, y)$ symmetric positiv definit. A suitable equvalence of norms follows by

$$\begin{aligned} a(u, u) &\geq \int_\Omega \rho \nabla u^T \nabla v = \rho |u|^2_{W^1}, \\ a(u, v) &\leq \max_{i,j} \|a_{i,j}\|_{L^\infty} |u|_{W^1} |v|_{W^1}. \end{aligned}$$

Prove this as a homework and formulate ta suitable weak form of a partial differential equation.

**Theorem 12** (Theorem of Rellich)**.** *Let $\Omega$ be a domain with Lipschitz-continuous boundary (e.g. assume that the boundary has a finite number of corners and edges). Then, $W^1(\Omega) \to L^2(\Omega)$ is an compact embedding. This means: Let $H > 0$ be a fixed number and $(x_n)_{n\in\boldsymbol{N}}$ a subset of $W^1(\Omega)$ with $\|x_n\|_{W^1(\Omega)} \leq H$. Then, there exists a convergent subsequence of $(x_n)_{n\in\boldsymbol{N}}$ which converges with respect to $L^2(\Omega)$.*

A proof of this theorem for a simple domain is given in Section 4.9. This important theorem is the basis of

1. the second Poincare inequality,
2. finite element interpolation theory,
3. abstract eigenvalue problem in infinite dimensional spaces, which is a generalization of principal axis theorem in finite dimensional spaces.

**Theorem 13** (Second Poincare's Inequality)**.** *$|\cdot|_{w^1}$ and $\|\cdot\|_{w^1}$ are equivalent norms on $H = \{u \in W^1(\Omega) | \int_\Omega u d(x,y) = 0\}$.*

**Proof:**
We have to prove that there is a constant $C > 0$, such that

$$\|u\|_{L^2} \leq C|u|_{W^1} \qquad \forall u \in H.$$

Suppose the opposite: $\forall n \in \mathbb{N}\ \exists u_n \in H$, such that

$$1 = \|u_n\|_{L^2} > n|u_n|_{W^1}.$$

This implies $\|\frac{du_n}{dx}\|_{L^2} \leq |u_n|_{W^1} \leq \frac{1}{n}$. By the Theorem of Rellich $(u_n)$ contains a convergent subsequence in $L^2(\Omega)$:

$$(u_{n_k})_{k \in \mathbf{N}}.$$

Let

$$u = \lim_{k\to\infty} u_{n_k} \tag{4}$$

in $L^2(\Omega), u \in L^2(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$. Then, we get

$$\begin{aligned}
|\int_\Omega u \frac{d\varphi}{dx} d(x,y)| &= |\lim_{k\to\infty} \int_\Omega u_{n_k} \frac{d\varphi}{dx} d(x,y)| \\
&= |\lim_{k\to\infty} \int_\Omega \frac{du_{n_k}}{dx} \varphi d(x,y)| \\
&\leq \lim_{k\to\infty} \|\frac{du_{n_k}}{dx}\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\leq \lim_{k\to\infty} \frac{1}{n_k} \|\varphi\|_{L^2(\Omega)}.
\end{aligned}$$

This implies $\frac{du}{dx} = 0$. Analogously, we get $\frac{du}{dy} = 0$. Thus, $u$ is constant. By (4) and $\int_\Omega u_{n_k} = 0$ we conclude $u = 0$. This leads to the contracdiction

$$1 = \lim_{k\to\infty} \|u_{n_k}\|_{L^2}^2 = \|u\|_{L^2}^2 = 0.$$

□

Using this inequality one can prove the existence and uniquness of Poisson's problem with pure Neumann boundary condition:

**Poisson's problem with Neumann boundary condition**
Let $H = \{u \in W^1 | \int u = 0\}$ and $f \in H$. Find $u \in H$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \qquad \forall v \in H.$$

One can prove that this weak solution of Possion'problem has the boundary condition

$$\frac{du}{dn} = 0 \quad \big|\partial\Omega.$$

## 4.3 Abstract Eigenvalue Problem

Let us assume that $(X, a)$ and $(Y, \langle,\rangle)$ are Hilbert spaces, such that

$$X \hookrightarrow Y$$

is a compact and dense embedding. (Embedding means that the mapping is injective, linear and continuous.) Assume that $X$ is an infinite dimensional vector space. Let us consider the eigenvalue problem : Find $u \in X, \lambda \in \mathbb{C}$ such that

$$a(u, v) = \lambda\langle u, v\rangle \quad \forall v \in X.$$

**Theorem 14.** *Then, there exists an infinite number of eigenvalues* $(\lambda_i)_{i\in\mathbb{N}}$ *with eigenvectors* $u_i$, $\|u_i\| = 1$ *such that*

- $0 < \lambda_1 \leq \lambda_2 \leq ... \to \infty$

  *This sequence contains only finite multiplicities.*

- *$(u_i)_{i\in\mathbb{N}}$ are $a-$ and $\langle .,.\rangle$-orthogonal. $(u_i)_{i\in\mathbb{N}}$ is a Hilbert space basis of $Y$.*

- *The following equation holds:*

$$\lambda_1 = \min_{v\in X} \frac{a(v,v)}{\langle v,v\rangle}.$$

**Example 21.** Consider the eigenvalue problem:

$$\begin{aligned} -\triangle u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

In case of $\Omega = ]0,1[^2$, the set of eigenvectors and eigenvalues is

$$u_{ij} = \sin(i\pi x)\sin(j\pi y), \quad \lambda_{ij} = \pi^2(i^2+j^2), \quad i,j \in \mathbb{N}.$$

**Example 22.** Consider the heat problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle u &= 0 \quad \text{in } \Omega, \\ u(t) &= 0 \quad \text{on } \partial\Omega,\ \forall t \geq 0, \\ u(0) &= u_0, \end{aligned}$$

where $u_0 \in \mathring{W}^1(\Omega)$ is a given function.

Then, a Fourier-analysis with respect to the general eigenvectors of $-\triangle$ implies that

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{5}$$

Let us prove (5). To apply Theorem 14, we choose

$$\begin{aligned} (X,a) &= \left(\mathring{W}^1(\Omega), (u,v) \mapsto \int_\Omega \nabla u \nabla v \; d(x,y)\right), \\ (Y,<>) &= \left(L^2(\Omega), (u,v) \mapsto \int_\Omega uv \; d(x,y)\right). \end{aligned}$$

By Theorem 12 and Theorem 11, we can apply Theorem 14. Let $(e_n)_{n\in\mathbb{N}}$ be the corresponding eigenvectors wit eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Since $(e_n)_{n\in\mathbb{N}}$ is a Hilbert space basis we can write

$$u_0 = \sum_{n\in\mathbb{N}} e_n c_n.$$

Then,
$$u(t) = \sum_{n \in \mathbb{N}} e_n c_n e^{-\lambda_n t}$$
is the solution of the heat equation. Let $\epsilon > 0$ be given. First choose $q \in \mathbb{N}$ such that
$$\sum_{n \geq q} |c_n|^2 < \frac{\epsilon^2}{2}.$$
Then, choose $t'$ such that
$$\sum_{n<q} \left|c_n e^{-\lambda_n t}\right|^2 < \frac{\epsilon^2}{2}$$
for every $t > t'$. This implies
$$\|u(t)\|_{L^2} = \left\| \sum_{n \in \mathbb{N}} e_n c_n e^{-\lambda_n t} \right\|_{L^2} < \epsilon$$
for every $t > t'$.

## 4.4 Sobolev Spaces for Periodic Functions

**Definition 22.**

$$L^2_{periodic}([-\pi,\pi[) \quad := \quad \{u \in \mathcal{F}(\mathbb{R}) \mid u|_{]-\pi,\pi[} \in L^2(]-\pi,\pi[) \quad \textit{and} \quad u(x) = u(x+2\pi)\ \forall x \in \mathbb{R}\}$$

*Let $s \in \mathbb{N}$. Then, define*

$$W^s_{periodic}([-\pi,\pi[) \quad := \quad \{u \in L^2_{periodic} \mid \frac{\partial^n u}{\partial x^n} \in L^2_{periodic}([-\pi,\pi[) \quad \forall n \leq s\}$$

*Here, $\frac{\partial u}{\partial x} \in L^2_{periodic}([-\pi,\pi[)$ means that there is a $g \in L^2_{periodic}([-\pi,\pi[)$ such that*

$$\int_{-\pi}^{\pi} u \frac{\partial \varphi}{\partial x}\, dx = -\int_{-\pi}^{\pi} g\varphi\, dx \quad \forall \varphi \in \mathcal{C}^\infty(\mathbb{R}) \cap L^2_{periodic}([-\pi,\pi[).$$

Analogously, define the Sobolev Space $L^2_{\text{periodic}}([-\pi,\pi[^d)$, for $d \in \mathbb{N}$.

## 4.5 Fractional Sobolev Spaces for Periodic Functions

The set of functions

$$\frac{1}{(\sqrt{2\pi})^d} e^{i\vec{n}\vec{x}} = B_{\vec{n}}(\vec{x}), \qquad \vec{n} \in \mathbb{Z}^d$$

forms a Hilbert-space basis of $L^2_{\mathbb{C}}([-\pi, \pi[^d)$.

**Theorem 15.** *Let* $u = \sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} B_{\vec{n}}(x)$. *Then,*

$$\frac{\partial u}{\partial x} \in L^2_{periodic} \quad \Leftrightarrow \sum_{n\in\mathbb{Z}} \|a_{\vec{n}} \cdot \vec{n}\|_2^2 < \infty.$$

*Furthermore, we get*

$$u \in W^k_{periodic}([-\pi, \pi[^d) \quad \Leftrightarrow \sum_{\vec{n}\in\mathbb{Z}^d} |a_{\vec{n}}|^2 (1 + \|\vec{n}\|_2^{2k}) < \infty$$

*for every* $k \in \mathbb{N}_0$.

Proof: Let us show this theorem for $k$ in 1D.
Let us assume $\frac{\partial u}{\partial x} \in L^2_{\mathbb{C}}([-\pi, \pi[)$. This means

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial x} \varphi(x) dx = -\int_{-\pi}^{\pi} \frac{\partial \varphi}{\partial x} u(x) dx$$

for every $\varphi \in \mathcal{C}_0^\infty(]-\pi, \pi[)$. Now, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\partial \varphi}{\partial x} u(x)\; dx &= \sum_{n\in\mathbb{Z}} a_n \int_{-\pi}^{\pi} B_n(x) \frac{\partial \varphi}{\partial x}\; dx \\
&= -\sum_{n\in\mathbb{Z}} a_n\, i\, n \int_{-\pi}^{\pi} B_n(x) \varphi\; dx
\end{aligned}$$

Let $b_n$ be the Fourier coefficients of $\frac{\partial u}{\partial x}$. Then, we obtain

$$\sum_{n\in\mathbb{Z}} b_n \int_{-\pi}^{\pi} B_n(x) \varphi\; dx \;=\; \sum_{n\in\mathbb{Z}} a_n\, i\, n \int_{-\pi}^{\pi} B_n(x) \varphi\; dx$$

Let us choose a sequence of functions in $\mathcal{C}_0^\infty(]-\pi,\pi[)$ which converges in $L^2(]-\pi,\pi[)$ to $B_m$. By this sequence, we get

$$b_n = a_n\, i\, n.$$

This implies that

$$\sum_{n\in\mathbb{Z}} |a_n|^2 n^2 < \infty.$$

End of proof.

**Definition 23.** *Let $u = \sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} B_{\vec{n}}(x)$ and $s > 0$ a real positive number. Then, define*

$$u \in H^s_{periodic}([-\pi,\pi[^d) \quad :\Leftrightarrow \sum_{\vec{n}\in\mathbb{Z}^d} |a_{\vec{n}}|^2 (1 + \|\vec{n}\|_2^{2s}) < \infty.$$

*$H^s_{periodic}$ is a Hilbert space with scalar product*

$$\left\langle \sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} B_{\vec{n}}, \sum_{\vec{n}\in\mathbb{Z}^d} b_{\vec{n}} B_{\vec{n}} \right\rangle := \sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} \bar{b}_{\vec{n}} (1 + \|\vec{n}\|_2^{2s}).$$

## 4.6 Trace Theorem

Let $\Omega \subset \mathbb{R}^d$ a bounded domain with piecewise continuous differentiable boundary. This means, that the boundary can be described by mappings

$$\varphi : [0,1] \to \partial\Omega,$$

where $\varphi$ is continuous and piecewise continuous differentiable. Then, there are different ways to define Sobolev spaces $W^p(\Omega)$ and $W^p(\partial\Omega)$, where $p \geq 0$.

**Theorem 16.** *Let us assume that $\Omega$ is bounded and the boundary of $\Omega$ is smooth enough. Furthermore, assume $s \geq \frac{1}{2}$. Then, there exists a linear and continuous mapping*

$$T : W^s(\Omega) \to W^{s-\frac{1}{2}}(\partial\Omega)$$

*and an extension operator*

$$F : W^{s-\frac{1}{2}}(\partial\Omega) \to W^s(\Omega)$$

*such that*

$$TF = Id.$$

*This implies that $T, F$ are linear and continuous and that*

$$T(\varphi)(x) = \varphi(x) \qquad \text{for } x \in \partial\Omega \text{ and } \varphi \in \mathcal{C}^\infty(\Omega) \cap W^s(\Omega).$$

**Example 23.** *Let us assume that $f \in L^2(\Omega)$ and $g \in W^{1.5}(\partial\Omega)$. Poisson's problem with inhomogeneous Dirichlet boundary conditions is:*

*Find $u \in W^1(\Omega)$ such that*

$$\begin{aligned} -\triangle u &= f \\ T(u) &= g. \end{aligned}$$

*To find the unique solution of this problem, consider the problem*

$$\begin{aligned} -\triangle w &= f + \triangle F(g) \\ T(w) &= 0. \end{aligned}$$

*Observe that $w \in H^1(\Omega)$ and $T(w) = 0$ is equivalent to $w \in \mathring{W}^1$. Then, the above homogenous Dirichlet boundary problem has a unique solution $w \in H^1(\Omega)$. Now, $u = w + F(g)$ is the solution of the inhomogeneous Dirichlet boundary problem.*

**Theorem 17.** *Let $s \geq 1$. Then, there exists a linear and continuous mapping*

$$T : H^s_{periodic}([-\pi, \pi[^2) \to H^{s-\frac{1}{2}}_{periodic}([-\pi, \pi[^1)$$

*such that*

$$\varphi(x, 0) = T(\varphi)(x) \quad \forall \varphi \in \mathcal{C}^\infty(\mathbb{R}^2) \cap L^2_{periodic}([-\pi, \pi[^2).$$

Proof: Let us prove the result for $s = 1$. For $a > 0$, we get

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{a + m^2} &\leq \int_0^\infty \frac{1}{a + x^2} dx = \\ &= \frac{1}{a} \int_0^\infty \frac{1}{1 + \left(\frac{x}{\sqrt{a}}\right)^2} dx = \frac{1}{a} \sqrt{a} \arctan \frac{x}{\sqrt{a}} \Big|_0^\infty \\ &= \frac{1}{a} \sqrt{a} \pi. = \frac{1}{\sqrt{a}} \pi. \end{aligned}$$

Choosing $a = 1 + n^2$, this implies

$$\sum_{m\in\mathbb{Z}} \frac{1}{1+n^2+m^2} \leq \frac{1}{\sqrt{1+n^2}} 5\pi. \tag{6}$$

Let $\varphi \in H^s_{\text{periodic}}([-\pi,\pi[^2) \cap \mathcal{C}(\mathbb{R}^2)$. Then, (see Heuser, Lehrbuch der Analysis) the Fourier sequence

$$\varphi(x,y) = \sum_{n,m\in\mathbb{Z}} a_{n,m} \frac{1}{2\pi} e^{i(nx+my)}$$

converges absolutly for every $(x, y)$. This implies that

$$\varphi(x,0) = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi} e^{inx} \sum_{m\in\mathbb{Z}} a_{n,m}$$

for every $x$. This implies that

$$b_n = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} a_{n,m}$$

are the Fourier coefficients of $\varphi(x, 0)$.

Now, let us prove the inequality:

$$\begin{aligned} \|\varphi(x,0)\|^2_{H^{1-\frac{1}{2}}} &= \sum_{n\in\mathbb{Z}} (1 + n^{2(s-\frac{1}{2})})|b_n|^2 \\ &\leq C \sum_{n,m\in\mathbb{Z}} (1 + (n^2+m^2)^s)|a_{n,m}|^2 = C\|\varphi\|^2_{H^1} \end{aligned} \tag{7}$$

for $s = 1$. Using (6), this inequality follows by

$$\sum_{n\in\mathbb{Z}} (1 + |n|^1)|b_n|^2 =$$

$$\begin{aligned} &= \sum_{n\in\mathbb{Z}} (1+|n|^1) \frac{1}{\sqrt{2\pi}^2} \left( \sum_{m\in\mathbb{Z}} a_{n,m} \sqrt{1+n^2+m^2} \left(\sqrt{1+n^2+m^2}\right)^{-1} \right)^2 \leq \\ &\leq \sum_{n\in\mathbb{Z}} (1+|n|) \frac{1}{2\pi} \left( \sum_{m\in\mathbb{Z}} \left(1+n^2+m^2\right) |a_{n,m}|^2 \right) \left( \sum_{m\in\mathbb{Z}} \left(1+n^2+m^2\right)^{-1} \right) \leq \\ &\leq \frac{5\pi}{2\pi} \sum_{n,m\in\mathbb{Z}} (1+n^2+m^2)|a_{n,m}|^2 (1+|n|) \frac{1}{\sqrt{1+n^2}} \\ &\leq 8 \sum_{n,m\in\mathbb{Z}} (1 + (n^2+m^2)^1)|a_{n,m}|^2. \end{aligned}$$

In the last eniquality, we applied the formula

$$\frac{1+n}{\sqrt{1+n^2}} \leq 2 \Leftarrow 1+2n+n^2 \leq 2(1+n^2) \Leftarrow 0 \leq 1-2n+n^2 = (1-n)^2.$$

Thus, we have proved (7) for every function $\varphi \in H^s_{\text{periodic}}([-\pi,\pi[^2)$. Since $H^s_{\text{periodic}}([-\pi,\pi[^2) \cap \mathcal{C}^\infty(\mathbb{R}^2)$ is dense in $H^s_{\text{periodic}}([-\pi,\pi[^2)$, (7) holds for every $u \in H^s_{\text{periodic}}([-\pi,\pi[^2)$.

End of proof.

## 4.7 Symmetric Extension

**Definition 24.** *Let* $\Omega = ]0,\pi[^2$ *and* $T = ]-\pi,\pi[^2$. *Then let us define the extension operator*

$$\begin{aligned} \tilde{}: L^2(\Omega) &\rightarrow L^2(T) \\ u &\mapsto \tilde{u}(x,y) = \left\{ \begin{array}{rl} u(x,y) & \textit{if } \; x,y \geq 0 \\ -u(x,y) & \textit{if } \; x \geq 0, y \leq 0 \\ -u(x,y) & \textit{if } \; x \leq 0, y \geq 0 \\ u(x,y) & \textit{if } \; x,y \leq 0 \end{array} \right\} \\ &= \operatorname{sgn}(x,y) \cdot u(|x|,|y|). \end{aligned}$$

Let $f \in L^2(\Omega)$. Let us consider the two problems:

- Find $u \in \mathring{W}^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \; d(x,y) = \int_\Omega fv \; d(x,y) \quad \forall v \in \mathring{W}^1(\Omega).$$

- Find $u' \in H^1_{\text{periodic}}$ such that

$$\int_T \nabla u' \nabla v \; d(x,y) = \int_T \tilde{f} v \; d(x,y) \quad \forall v \in H^1_{\text{periodic}}(T).$$

Observe that $\tilde{u} = u'$.

## 4.8 Regularity of Elliptic Equations

**Definition 25.** *Let $\Omega \subset \mathbb{R}^2$ a bounded domain. Poisson's equation with Dirichlet b.c. is called $W^2$-regular, if for every $f \in L^2(\Omega)$, there exists a $u \in W^2(\Omega)$ such that*

$$\begin{aligned} -\triangle u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

**Theorem 18.** *Let $\Omega \subset \mathbb{R}^2$ a bounded domain with piecewise continuous differentiable boundary. Furthermore assume that all interior angles of the boundary are smaller than $\pi$. Then, Poisson's equation is $W^2$-regular.*

**Theorem 19.** *Let $T = ]-\pi, \pi[^2$. Poisson's equation is $H^2_{periodic}$-regular in the following sense:*

*For every $f \in L^2_{periodic}(T)$, $\int_T f = 0$, there exists a $u \in H^2_{periodic}(T)$ such that*

$$\begin{aligned} -\triangle u &= f \quad \text{on } \mathbb{R}^2 \\ \int_T u &= 0. \end{aligned}$$

Proof: Let $f = \sum_{n,m\in\mathbb{Z}} a_{n,m} B_{n,m}$. Since $\int_T f = 0$, we get $a_{0,0} = 0$.
Now, define

$$b_{n,m} = \begin{cases} 0 & \text{if } n = m = 0 \\ \frac{a_{n,m}}{n^2+m^2} & \end{cases}$$

Then, $u = \sum_{n,m\in\mathbb{Z}} b_{n,m} B_{n,m}$ is the solution of Poisson's equation. Since

$$\begin{aligned} \|u\|^2_{H^2} &= \sum_{n,m\in\mathbb{Z}} (1 + (n^2+m^2)^2)|b_{n,m}|^2 = \sum_{n,m\in\mathbb{Z}} (1 + (n^2+m^2)^2)|a_{n,m}|^2 \frac{1}{(n^2+m^2)^2} \\ &\leq 2 \sum_{n,m\in\mathbb{Z}} |a_{n,m}|^2 = \|f\|^2_{L^2}. \end{aligned}$$

End of proof.

By the extension concept in Section 4.7, we get the following theorem

**Theorem 20.** *Let $\Omega = ]0, \pi[^2$. Then, Poisson's equation with Dirichlet boundary conditions is $W^2$-regular.*

## 4.9 Proof of Compact Embedding

**Theorem 21.** *$H^1_{periodic}(]-\pi,\pi[)$ is compactly embedded in $L^2_{periodic}(]-\pi,\pi[)$.*

Proof:

Let $u_n = \sum_{k\in\mathbb{Z}} a^n_k B_k$ be a sequence such that $\|u_n\|_{H^1} \leq 1 \;\; \forall n \in \mathbb{N}$. Then, the following inequality holds

$$\sum_{k\in\mathbb{Z}} |a^n_k|^2(1+k^2) \leq 1.$$

This implies $|a^n_k|^2 \leq \frac{1}{1+k^2} \;\; \forall n \in \mathbb{N}$.

Let us construct a subsequence as follows

- Choose subsequence such that

  $$\left(a_1^{n_s^1}\right)_{s\in\mathbb{N}}$$

  converges, where $(n^1_s)_{s\in\mathbb{N}}$ is a strictly monotonic increasing sequence.

- Choose a strictly monotonic increasing subsequence $(n^{l+1}_s)_{s\in\mathbb{N}}$ from the sequence $(n^l_s)_{s\in\mathbb{N}}$ such that

  $$\left(a_{l+1}^{n_s^{l+1}}\right)_{s\in\mathbb{N}}$$

  converges.

- Define $m_s = n^s_s$. Then, the sequence

  $$(a^{m_s}_k)_{s\in\mathbb{N}} = \left(a_k^{n_s^s}\right)_{s\in\mathbb{N}}$$

  converges for every $k$. Define

  $$b_k = \lim_{s\to\infty} a^{m_s}_k.$$

Let us define the function

$$w = \sum_{k\in\mathbb{Z}} b_k B_k.$$

This function is in $L^2(\Omega)$ since

$$\sum_{k\in\mathbb{Z}} |b_k|^2 \leq \sum_{k\in\mathbb{Z}} \frac{1}{1+k^2} \leq 10.$$

The proof of the theorem is complete, if we can show

$$\lim_{s\to\infty} \|w - u_{m_s}\|_{L^2(\Omega)} = 0. \tag{8}$$

To prove this convergence, let $\epsilon > 0$ be given. Now, choose $q \in \mathbb{N}$ such that

$$\sum_{|k|\geq q} \frac{1}{1+k^2} < \frac{\epsilon}{8}.$$

This implies

$$\sum_{|k|\geq q} |a_k^{m_s} - b_k|^2 \leq \sum_{|k|\geq q} \frac{4}{1+k^2} \leq \frac{\epsilon}{2}.$$

By construction, we get

$$\lim_{s\to\infty} \sum_{|k|<q} |a_k^{m_s} - b_k|^2 = 0.$$

This implies

$$\lim_{s\to\infty} \sum_{k\in\mathbb{Z}} |a_k^{m_s} - b_k|^2 = 0.$$

This completes the proof of (8).
End of Proof.

# 5 Distribution Theory

## 5.1 Basic Theory

**Definition 26.** *A set $K \subset \mathbb{R}^d$ is compact if it is closed and bounded.*

**Definition 27** (Convergence of Test Functions)**.** *Let $\Omega \subset \mathbb{R}^d$ open and $\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega)$. Let $(\Phi_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$ and $\Phi \in \mathcal{D}(\Omega)$. Then, $(\Phi_k)_{k\in\mathbb{N}}$ is called to be convergent to $\Phi \in \mathcal{D}(\Omega)$, if the following properties hold:*

1. *There is a compact set $K \subset \Omega$ such that $\mathrm{supp}(\Phi_k) \subset K$.*
2. *$\lim_{k\to\infty} \|D^\alpha\Phi_k - D^\alpha\Phi\|_\infty = 0$ for every multi-index $\alpha$.*

*Furthermore, let us write*

$$\Phi_k \overset{\mathcal{D}}{\to} \Phi$$

*if* $(\Phi_k)_{k\in\mathbb{N}}$ *converges to* $\Phi \in \mathcal{D}(\Omega)$.

**Definition 28.** *A linear mapping* $T : \mathcal{D}(\Omega) \to \mathbb{R}$ *is called distribution, if*

$$\Phi_k \overset{\mathcal{D}}{\to} \Phi \qquad \Rightarrow \qquad T(\Phi_k) \overset{k\to\infty}{\longrightarrow} T(\Phi).$$

*Let* $\mathcal{D}'$ *be the set of all distributions.*

**Example 24.** *Let* $f : \Omega \to \mathbb{R}$ *be a local integrable function. This means that* $\int_K |f(x)|dx < \infty \quad \forall K \subset \Omega$ *compact.*
*Then, the distribution corresponding to* $f$ *is:*

$$\begin{aligned} T_f : \Phi &\mapsto \int_\Omega f\Phi dx, \\ T_f(\Phi) &= \int_\Omega f\Phi dx. \end{aligned}$$

*Therefore, distrubution are called generalized functions!*

Let us prove that $T_f$ is a distribution. First, we have to show that $\int_\Omega |f\Phi|dx < \infty$:

$$\int_\Omega |f\Phi|dx = \int_{\text{supp}(\Phi)\subset\Omega} |f\Phi|dx \le \overbrace{\int_{\text{supp}(\Phi)} |f(x)|dx}^{<\infty} \cdot \overbrace{\|\Phi\|_\infty}^{<\infty}$$

Now, assume $\Phi_k \overset{\mathcal{D}}{\to} \Phi$. Then, we get

$$\begin{aligned} \lim_{k\to\infty} |T_f(\Phi_k) - T_f(\Phi)| &\le \lim_{k\to\infty} |\int_\Omega f\Phi_k dx - \int_\Omega f\Phi dx| \\ &\le \lim_{k\to\infty} |\int_K (f\Phi_k - f\Phi)dx| \\ &\le \lim_{k\to\infty} \left|\int_K f dx\right| \|\Phi_k - \Phi\|_\infty = 0, \end{aligned}$$

where $K$ compact such that $\text{supp}(\Phi_k) \subset K \subset \Omega$. $\square$

**Example 25.** *The delta distribution is:*

$$\begin{aligned} \delta : \mathcal{D} &\to \mathbb{R} \\ \Phi &\mapsto \Phi(0). \end{aligned}$$

Homework: Prove that $\delta$ is a distribution!

**Formula 4.**

$$\begin{aligned} T+G &\in \mathcal{D}', \\ \lambda T &\in \mathcal{D}' \end{aligned}$$

*for every $T, G \in \mathcal{D}'$ and $\lambda \in \mathbb{R}$.*

**Definition 29.** *Let $T \in \mathcal{D}'$, then*

$$\begin{aligned} \frac{dT}{dx} : \mathcal{D} &\to \mathbb{R} \\ \Phi &\mapsto -T\left(\frac{d\Phi}{dx}\right) \end{aligned}$$

*is defined to be the derivative of $T$.*

Homework: Show that $\frac{dT}{dx}$ is a distribution.

**Example 26.** *Let $f \in C^1(\Omega)$. Then*

$$\frac{dT_f}{dx} = T_{\frac{df}{dx}}.$$

Proof:

$$\begin{aligned} \frac{dT_f}{dx}(\Phi) &= -T_f(\frac{d\Phi}{dx}) \\ &= -\int_\Omega f\frac{d\Phi}{dx}\,dx = \int_\Omega \frac{df}{dx}\Phi\,dx \\ &= T_{\frac{df}{dx}}(\Phi) \qquad \forall \Phi \in \mathcal{D}(\Omega). \end{aligned}$$

□

**Example 27.** *Let*

$$H(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ 1 & \text{for} \quad x > 0 \end{cases}.$$

*Then, the following formula holds*

$$\frac{dH}{dx} = \delta.$$

Proof:

$$\begin{aligned}\frac{dH}{dx}(\Phi) &= -H\left(\frac{d\Phi}{dx}\right) = -\int_{-\infty}^{\infty} H\frac{d\Phi}{dx}\,dx \\ &= -\int_0^{\infty}\frac{d\Phi}{dx} = -\Phi\big|_0^{\infty} \\ &= \Phi(0) = \delta(\Phi).\end{aligned}$$

□

**Definition 30.** *Let $T \in \mathcal{D}'$ and $T_n \in \mathcal{D}', n \in \mathbb{N}$ be a sequence of distributions. Then, $(T_n)_{n\in\mathbb{N}}$ converges to $T$, if*

$$\lim_{n\to\infty} T_n(\Psi) = T(\Psi)$$

*for every $\Psi \in \mathcal{D}$. Let us write $\lim_{n\to\infty} T_n = T$.*

Remark: Let $\mathcal{M}$ be the mollifier function in Example 16. Then, define

$$T_n := T_{(n\mathcal{M}(\cdot * n))}$$

One can show that

$$\lim_{n\to\infty} T_n = \delta.$$

## 5.2 Convolution and Applications

**Definition 31.** *Convolution,(German: Faltung) Let $T \in \mathcal{D}'$ und $\varphi \in \mathcal{D}$. Then the convolution is defined by*

$$(T * \varphi)(x) = T(\varphi(x - \cdot)).$$

**Example 28.** *Let $f : \Omega \to \mathbb{R}$ be a local integrable function. Then,*

$$\begin{aligned}(T_f * \varphi)(x) &= T_f(\varphi(x - \cdot)) \\ &= \int_\Omega f(y)\varphi(x-y)dy.\end{aligned}$$

*This shows that the convolution of distrubutions (generalized functions) generalizes the concept of convolutions for classical functions!*

**Formula 5.** *Let $T$ be a distrubution on $\mathbb{R}^d$ and $\varphi \in \mathcal{D}$. Then, the following formulas hold:*

$$T * \varphi \in \mathcal{C}^\infty(\mathbb{R}^d).$$

$$D^\alpha(T * \varphi) = D^\alpha T * \varphi = T * D^\alpha \varphi.$$

*Let $\delta$ be the delta distribution. Then,*

$$\delta * \varphi = \varphi.$$

Let us prove only the last formula:

$$(\delta * \varphi)(x) = \delta(y \longmapsto \varphi(x-y)) = \varphi(x-0) = \varphi(x).$$

□

**Definition 32.** *Let $D = \sum_{s=0}^m a_s \frac{d^s}{dx^s}$, $a_s \in \mathbb{C}$, be a differential operator. $F \in \mathcal{D}'$ is called fundamental solution, if $DF = \delta$.*

**Theorem 22.** *Let $F \in \mathcal{D}'$ be a fundamental solution of the differential operator $D = \sum_{s=0}^m a_s \frac{d^s}{dx^s}$, $a_s \in \mathbb{C}$. Furthermore, let $f \in \mathcal{D}$.*
*Then, $u := F * f$ is a solution of the equation*

$$Du = f.$$

Proof:

$$Du = D(F * f) = DF * f = \delta * f = f.$$

□

**Example 29.** *Consider the differential equation*

$$\frac{d^2}{dx^2} u = f,$$

*where $f \in \mathcal{D}$. The fundamental solution is:*

$$F = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ x & \text{for} \quad x > 0 \end{cases},$$

*since $\frac{d^2}{dx^2} F = \frac{d}{dx} H = \delta$, where*

$$H = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ 1 & \text{for} \quad x > 0 \end{cases}.$$

*Thus, a solution of the above differential equation is obtained by*

$$\begin{aligned} u(x) = (F * f)(x) &= \int_{-\infty}^{\infty} F(y) f(x-y) dy \\ u(x) &= \int_{0}^{\infty} y f(x-y) dy. \end{aligned}$$

Literature see [1], [2], [7], [4], [5], [6], [3].

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